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SPECTRAL PROBLEM FOR SHELLS WITH FLUID

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Variational eigenvalue equations describing vibrations of orthotropic shells containing an ideal incompressible fluid are obtained. The vibration frequencies are assumed to be small, which makes it possible to use linear equations and to consider the boundary of the wet surface of the shell to be unchanged. The equations of anisotropic shells are based on the linear relations of multifield theory, which allows to obtain a more accurate model of anisotropic shells that satisfies the conditions of the finite-element method. The fluid flow is considered irrotational and is described using the Laplace equation. A finite-element algorithm is designed to determine the natural frequencies and modes of vibrations of an arbitrary multilayer orthotropic shell of revolution which is partially filled with an ideal incompressible fluid.

Key words: *shell theory, ideal fluid, vibration theory.*

Introduction. The presence of tanks partially filled with fluid is an integral part of many structures of various applications. These are various reservoirs, fuel tanks, etc. Recently, there has been an increased use of composite materials with anisotropic mechanical properties in the manufacture of such structures. The filling of a structure with fluid changes its dynamic properties significantly. Therefore, there is an urgent need to study the vibrations of shells containing fluids in internal cavities.

Equation of Motion for a Shell of Revolution. Anisotropic composite materials of low shear rigidity are widely used to manufacture shells. The most adequate model for the behavior of such shells is six-modal theory based on multifield hypotheses, in which assumptions on displacement distributions are supplemented by independent assumptions on transverse strains [1].

Let us consider an orthotropic shell of revolution in a coordinate system attached to the surface Ω (Fig. 1). According to the adopted model, the displacement vector components of the points of the shell are written in the global cylindrical coordinates r, x, φ as follows:

$$
\begin{gathered}
U_x(s,\varphi,z) = u_x(s,\varphi) + z(\psi_1(s,\varphi)\sin\theta - \chi(s,\varphi)\cos\theta),\\
U_r(s,\varphi,z) = u_r(s,\varphi) + z(\psi_1(s,\varphi)\cos\theta + \chi(s,\varphi)\sin\theta), \quad U_\varphi(s,\varphi,z) = u_\varphi(s,\varphi) + z\psi_2(s,\varphi).
\end{gathered} \tag{1}
$$

Here s and φ are the Gaussian coordinates of the surface of reference, z is the normal coordinate, θ is the slope of the normal to the coordinate surface to the axis of rotation of the shell, u_r, u_x, and u_φ are the displacements of the points of the coordinate surface, and ψ_1 and ψ_2 are the angles of rotation and χ is the elongation of the element normal to the surface. From the decomposition (1) using the kinematic Cauchy relations, we obtain the strain-tensor components $2e_{ij} = (U_{i,j} + U_{j,i})$. The components e_{13} and e_{23} calculated from these formulas do not ensure satisfaction of the boundary conditions on the stresses on the boundary surfaces of the shell and introduce a certain error to the calculation results.

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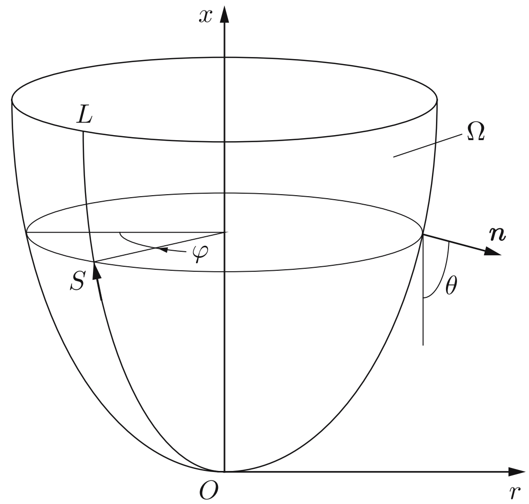

Fig. 1. Geometry of the shell.

Therefore, according to the concept adopted in [1], independent hypotheses on transverse strains are formulated in addition to the kinematic hypotheses (1). We shall seek transverse shear strains in the form

$$2e_{\alpha 3}(s,\varphi,z) = 4(z/h)(1-z/h)\gamma_{\alpha 3}(s,\varphi) \qquad (\alpha = 1,2), \tag{2}$$

where h is the shell thickness. Here the inner surface of the shell is chosen as the surface of reference. In our case, this form satisfies the condition of zero strains and shear stresses on the boundary surfaces of the shell, i.e.,

$$e_{\alpha 3}(s,\varphi,0) = e_{\alpha 3}(s,\varphi,h) = 0, \qquad \tau_{\alpha 3}(s,\varphi,0) = \tau_{\alpha 3}(s,\varphi,h) = 0 \qquad (\alpha = 1,2).$$

The stress-tensor components τ_{ij} obey Hooke's law for orthotropic media. The strain factor $\gamma_{\alpha 3}$ is determined from the condition of the least standard deviation of the transverse shear strain along the shell thickness obtained from the kinematic hypotheses (1) and the transverse strain hypothesis (2):

$$\min \int\limits_0^h \Big\{ 4\,\frac{z}{h}\Big(1-\frac{z}{h}\Big)\gamma_{\alpha 3}(s,\varphi) - (U_{i,j}+U_{j,i}) \Big\}^2 dz \qquad (\alpha = 1,2).$$

To derive the equation of motion for the shell, we use the principle of possible displacements and supplement it, according to the d'Alembert principle, by the work of inertial forces

$$\delta(E - A - T) = 0. \tag{3}$$

Here E is the potential strain energy, A is the work of external forces, and T is the work of inertial force. Let us consider the variation in the potential strain energy of the system

$$\delta E = \int\limits_V \tau_{ij}\delta e_{ij}\, dV = \int\limits_0^L \int\limits_0^{2\pi} \int\limits_0^h \tau_{ij}\, \delta e_{ij}\, dz\, r\, d\varphi\, ds, \tag{4}$$

where V is the volume of the shell body and L is the length of the shell meridian. Here by virtue of the adopted hypotheses, the stress- and strain-tensor components bear a known relation to the coordinate z. Integrating over the thickness in (4), we obtain

$$\delta E = \int\limits_0^L \int\limits_0^{2\pi} \{T\}^{\mathrm{t}} \delta\{\varepsilon\}\, r\, d\varphi\, ds, \tag{5}$$

where $\{\varepsilon\} = \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}, \varepsilon_{13}, \varepsilon_{23}, \varepsilon_{33}, \varkappa_{11}, \varkappa_{22}, \varkappa_{12}, \varkappa_{21}, \varkappa_{13}, \varkappa_{23}\}^{\mathrm{t}}$ and $\{T\} = \{T_{11}, T_{12}, T_{22}, T_{13}, T_{23}, T_{33}, M_{11}, M_{22}, M_{12}, M_{21}, M_{13}, M_{23}\}^{\mathrm{t}}$ are the generalized-strain and -stress vectors, ε_{ij} is the tension and shear, $\varkappa_{ij}$ is the bending and torsion of the coordinate surface, T_{ij} are the tensile and shear forces, and M_{ij} are the bending and torsional moments reduced to the coordinate surface. The generalized-stress vector is related to the generalized-strain vector by Hooke's law in the form

$$\{T\} = [D]\{\varepsilon\},$$

where $[D]$ is the symmetric rigidity matrix [2].

Let us write the variation in the work of inertial forces:

$$\delta T = \int_0^L \int_0^{2\pi} \int_0^h \{U\}^{\mathrm{t}} \rho\, \delta\{\ddot{U}\}\, dz\, r\, d\varphi\, ds.$$

Here the displacement vector $\{U\}$ is given by components in the global coordinates by decomposition (1) and ρ is the density of the shell material. Integration over the thickness yields

$$\delta T = \int_0^L \int_0^{2\pi} \{u\}^{\mathrm{t}} [\rho]\, \delta\{\ddot{u}\}\, r\, d\varphi\, ds, \tag{6}$$

where $\{u\} = \{u_r, u_x, u_\varphi, \psi_1, \chi, \psi_2\}^{\mathrm{t}}$ is the generalized-displacement vector and $[\rho]$ is the density matrix.

Let the shell considered be filed with an ideal fluid. Therefore, of the external forces we shall take into account only the normal pressure p_n. In this case, the work of external forces in (3) is defined by the work of pressure for the normal displacement of the inner surface of the shell w_n:

$$\delta A = \int_0^L \int_0^{2\pi} p_n\, \delta w_n\, r\, d\varphi\, ds = \int_0^L \int_0^{2\pi} p_n\, \delta(u_r \sin\theta - u_x \cos\theta)\, r\, d\varphi\, ds. \tag{7}$$

In the case of harmonic excitation for shells of revolution, the load and the required displacements can be written as

$$p_n(s, \varphi, t) = \mathrm{e}^{i\omega t} \sum_k p_n^k(s) \cos k\varphi,$$

$$U_r = \mathrm{e}^{i\omega t} \sum_k U_r^k \cos k\varphi, \quad U_x = \mathrm{e}^{i\omega t} \sum_k U_x^k \cos k\varphi, \quad U_\varphi = \mathrm{e}^{i\omega t} \sum_k U_\varphi^k \sin k\varphi$$

or for the generalized-displacement vector,

$$\{u(s, \varphi, t)\} = \mathrm{e}^{i\omega t} \sum_k \Big\{ u_r^k(s) \cos k\varphi, u_x^k(s) \cos k\varphi, u_\varphi^k(s) \sin k\varphi, \psi_1^k(s) \cos k\varphi, \chi^k(s) \cos k\varphi, \psi_2^k(s) \sin k\varphi \Big\}^{\mathrm{t}}. \tag{8}$$

Substituting (8) into (3) with allowance for (5)–(7) and integrating over the circumferential coordinate, we obtain the following variational equation for the kth harmonic:

$$\int_0^L \delta\{\varepsilon\}_k^{\mathrm{t}} [D] \{\varepsilon\}_k\, r\, ds - \omega^2 \int_0^L \delta\{u\}_k^{\mathrm{t}} [\rho] \{u\}_k\, r\, ds - \int_0^L \delta(u_r^k \sin\theta - u_x^k \cos\theta) p_n^k\, r\, ds$$

$$= \int_0^L \delta\{\varepsilon\}_k^{\mathrm{t}} [D] \{\varepsilon\}_k\, r\, ds - \omega^2 \int_0^L \delta\{u\}_k^{\mathrm{t}} [\rho] \{u\}_k\, r\, ds - \int_0^L \delta\{u\}_k^{\mathrm{t}} \{L\} p_n^k\, r\, ds = 0, \tag{9}$$

where $\{u\}_k$ is the kth harmonic of the generalized-displacement vector; $\{L\} = \{\sin\theta, -\cos\theta,\ 0, 0, 0, 0\}^{\mathrm{t}}$ is the expanded vector of the direction cosines of the normal to the shell surface.

Equation (9) is in essence the variational equation of forced steady-state vibrations of the shell of revolution loaded by a pressure which varies harmonically.

Equation of Motion for Ideal Fluid. We consider the case where the shell of revolution is partially filled with an ideal incompressible fluid. It is assumed that the vibrations of the shell are small and the fluid flow is irrotational in a stationary coordinate system. Let the axis of rotation x be directed vertically upward (Fig. 2).

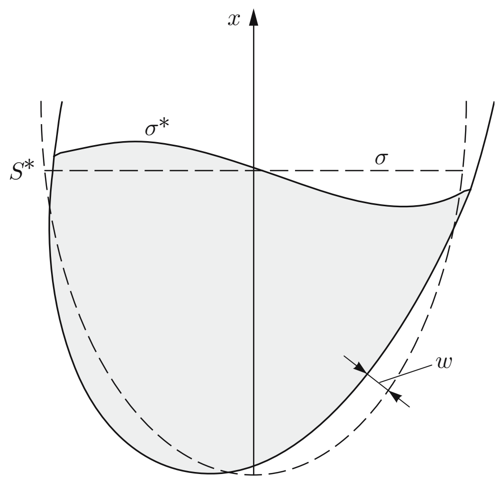

Fig. 2. Shell with fluid.

Then, the free surface of the fluid σ is described by the equation $x = \text{const}$, and the lateral surface S is determined by the wet part of the inner surface of the shell Ω. During motion of the system, the points belonging to the inner surface of the shell perform the displacements u_x, u_r, and u_φ and the free-surface boundary occupies the position σ^*. In this case, the fluid pressure has the dynamic component $p(x, r, \varphi, t)$.

By virtue of the above assumption on the smallness of the vibration frequency, the boundary of the wet surface of the shell can be considered unchanged during its motion. We denote by S^* the inner surface of the shell occupied by the fluid during motion. In the case of small vibrations, the boundary conditions can be extended from the surfaces S^* and σ^* to their neighboring surfaces S and σ. Then, the pressure p should satisfy the Laplace equation [3]

$$\nabla^2 p = 0 \tag{10}$$

with the boundary conditions

$$\frac{\partial p}{\partial \boldsymbol{n}} = -\rho^* \frac{\partial^2 w}{\partial t^2} \quad \text{on } S, \qquad \frac{\partial p}{\partial \boldsymbol{n}} + \frac{1}{g}\frac{\partial^2 p}{\partial t^2} = 0 \quad \text{on } \sigma. \tag{11}$$

Here ρ^* is the fluid density, $\boldsymbol{n}$ is the unit normal vector to the fluid surface, w is the normal component of the displacement vector, and g is the acceleration of gravity.

Equation (10) with boundary conditions (11) describes the dynamic equilibrium of the fluid in the case of disturbance of its lateral surface S. Let us consider the case where the normal displacement of the lateral surface w is a harmonic function of time:

$$w(s, \varphi, t) = \tilde{w}(s, \varphi)\, \mathrm{e}^{i\omega t}\,.$$

We seek a stationary solution of Eq. (10) of the form

$$p(x, r, \varphi, t) = \tilde{p}(x, r, \varphi)\, \mathrm{e}^{i\omega t}\,.$$

Then, the peak pressure values should satisfy the equation

$$\nabla^2 \tilde{p} = 0 \tag{12}$$

and the boundary conditions

$$\frac{\partial \tilde{p}}{\partial \boldsymbol{n}} - \rho^* \omega^2 \tilde{w} = 0 \quad \text{on } S, \qquad \frac{\partial \tilde{p}}{\partial \boldsymbol{n}} - \frac{\omega^2}{g}\,\tilde{p} = 0 \quad \text{on } \sigma. \tag{13}$$

The variational formulation of the problem involves a consideration of the functional

$$J = \frac{1}{2}\iiint\limits_V (\nabla \tilde{p})^2\, dV - \omega^2 \rho^* \iint\limits_S \tilde{w}\tilde{p}\, dS - \frac{\omega^2}{2g}\iint\limits_\sigma \tilde{p}^2\, d\sigma,$$

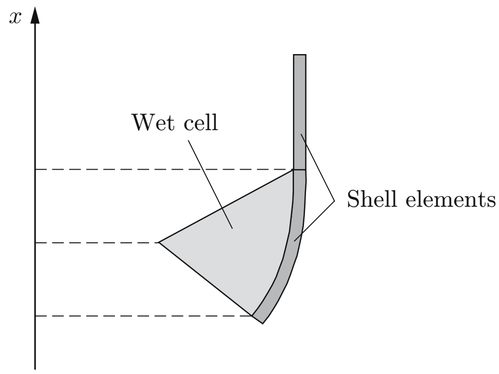

Fig. 3. Finite elements of the shell and fluid.

which takes a stationary value for the pressure $\tilde{p}$ which is a solution of problem (12), (13) [4]. A necessary condition for the stationarity of the functional J is the vanishing of its first variation:

$$\delta J = \iiint\limits_V \nabla\tilde{p}\,\delta(\nabla\tilde{p})\,dV - \omega^2\rho^* \iint\limits_S \tilde{w}\,\delta\tilde{p}\,dS - \frac{\omega^2}{g}\iint\limits_\sigma \tilde{p}\,\delta\tilde{p}\,d\sigma = 0. \tag{14}$$

Let us consider the case where the normal displacement of the boundary $\tilde{w}$ can be represented by a Fourier series in the circumferential coordinate:

$$\tilde{w}(s,\varphi) = \sum_k \{L\}^{\mathrm{t}}\{u\}_k \cos k\varphi. \tag{15}$$

Here $\{u\}_k$ and $\{L\}$ are the displacement vectors and the vectors of the direction cosines of the normal [formulas (8) and (9)]. We seek a solution of the variational equation (14) in the form

$$\tilde{p}(x,r,\varphi) = \sum_k p_k(x,r)\cos k\varphi. \tag{16}$$

Taking into account (15) and (16), we write the variational equation (14) in cylindrical coordinates and integrate it over φ. In this case, the system is split into independent equations for each circumferential harmonic number k

$$\int\limits_x\int\limits_r \Big(\frac{\partial p_k}{\partial r}\,\delta\,\frac{\partial p_k}{\partial r} + \frac{\partial p_k}{\partial x}\,\delta\,\frac{\partial p_k}{\partial x} + k^2 p_k\,\delta p_k\Big)\,r\,dr\,dx$$

$$- \omega^2\rho^* \int\limits_0^{s_l} (u_r^k \sin\theta - u_x^k\cos\theta)\,\delta p_k\,r\,ds - \frac{\omega^2}{g}\int\limits_0^R p_k\,\delta p_k\,r\,dr = 0. \tag{17}$$

Here s_l is the boundary of the shell segment wet with the fluid (see Fig. 2) and R is the radius of the fluid free surface. Equation (17) can be written in matrix form

$$\int\limits_x\int\limits_r \delta\{\gamma\}_k^{\mathrm{t}}\{\gamma\}_k\,r\,dr\,dx - \frac{\omega^2}{g}\int\limits_0^R \delta p_k\,p_k\,r\,dr - \omega^2\rho^*\int\limits_0^{s_l}\delta p_k\{L\}^{\mathrm{t}}\{u\}_k\,r\,ds = 0, \tag{18}$$

where $\{\gamma\} = \{\partial p_k/\partial x, \partial p_k/\partial r, kp_k/r\}^{\mathrm{t}}$ is the pressure gradient.

Thus, we obtained the variational problem of steady-state forced vibrations of the fluid under a small harmonic change in the tank shape.

Natural Vibrations of Shells Partially Filled with Fluid. Equations (9) and (18) are the variational equations of forced vibrations of a shell and fluid, respectively, for the kth circumferential harmonic of the Fourier series expansion of the solution in the circumferential coordinate. In the contact zone between these bodies, the normal displacements of points of the shell and the pressure acting on the shell coincide with the normal displacement of the fluid boundary and the pressure arising in it, i.e., $w_n = w$ and $p_n = p$. Combining these two equations,

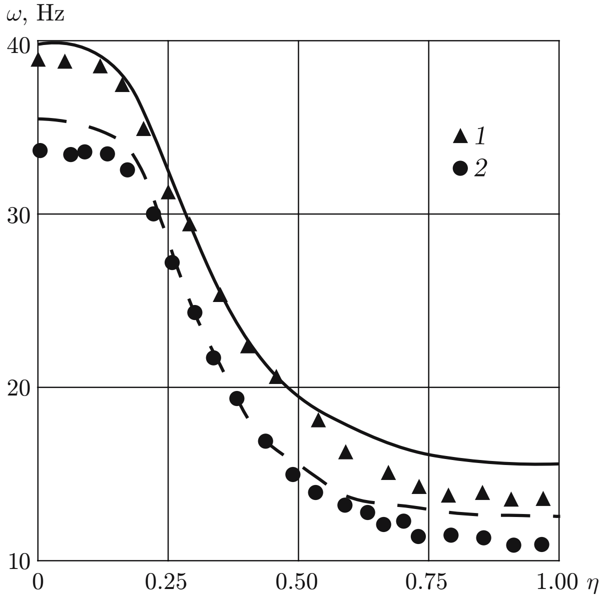

Fig. 4

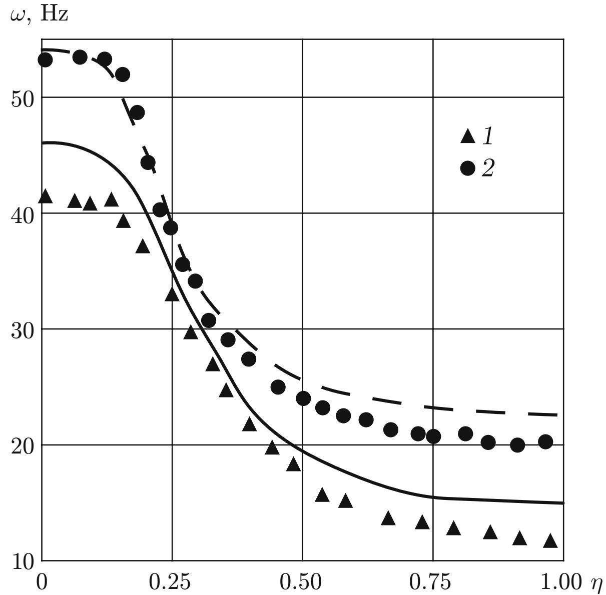

Fig. 5

Fig. 4. Natural frequencies of shell vibrations versus fluid level for the 3rd and 4th circumferential harmonics: the points refer to the experimental data of [5] for the 3rd harmonic (1) and the 4th harmonic (2); the curves refer to calculations for the 3rd harmonic (solid curve) and the 4th harmonic (dashed curve).

Fig. 5. Natural frequencies of shell vibrations versus fluid level for the 5th and 6th circumferential harmonics: the points refer to the experimental data of [5] for the 5th harmonic (1) and the 6th harmonic (2); the curves refer to calculations for the 5th harmonic (solid curve) and 6th harmonic (dashed cure).

we obtain a homogeneous system that can be interpreted as the spectral problem of natural vibrations of a shell partially filled with an ideal fluid. This problem was solved using a finite element method. For this, the region occupied by the fluid was partitioned into triangular finite elements and the shell was represented by one-dimensional curvilinear ring-type finite elements [2] with a quadratic approximation. The elements on the contact boundary between the shell and the fluid were matched with each other (Fig. 3).

The displacements u_r, u_x, and u_φ and the pressure p are approximated by quadratic polynomials and ψ_1, ψ_2, and χ by first-order polynomials (here and below, the subscript k is omitted). We assume that $\{V\}$ and $\{q\}$ are the nodal-parameter vectors for the shell and fluid, respectively, $[N]$ and $[n]$ are the shape-function matrices for the shell and fluid, respectively, and $[b]$ is the pressure-gradient matrix. Then, we can write $\{u\} = [N]\{V\}$, $p = [n]\{q\}$, and $\{\gamma\} = [b]\{q\}$. Formally, combining the equations of the shell and the fluid, we obtain the following homogeneous system of linear algebraic equations:

$$([K_*] + [K] - [T] - \omega^2([M] + g^{-1}[R]))\{W\} = 0. \tag{19}$$

This equation uses the following notation: $\{W\} = \{V, q\}^{\mathrm{t}}$ is the generalized vector of the nodal parameters, $[T] = \begin{bmatrix} 0 & \omega^2\rho^*[P]^{\mathrm{t}} \\ [P] & 0 \end{bmatrix}$ is the matrix of the mutual effect of the fluid and the shell, $[K]$ and $[M]$ are the rigidity and mass matrices of the shell, $[P] = \sum\limits_{e=1}^{NE_c} \int\limits_S [N]^{\mathrm{t}}[L]^{\mathrm{t}}[n]\, r\, ds$ is the matrix of the effect of the fluid on the shell, $[K_*] = \sum\limits_{e=1}^{NE_l} \int\limits_x \int\limits_r [b]^{\mathrm{t}}[b]\, r\, dr\, dx$ is the gravitational-rigidity matrix of the fluid, $[R] = \sum\limits_{e=1}^{NE_{\mathrm{surf}}} \int\limits_r [n]^{\mathrm{t}}[n]\, r\, dr$ is the inertia matrix of the fluid free surface, NE_c is the number of shell elements adjoining the fluid, NE_l is the number of fluid elements, and NE_{surf} is the number of elements belonging to the fluid free surface. The matrices included in (19) are matched to the structure of the generalized nodal-parameter vector $\{W\}$.

Equation (19) is treated as a generalized eigenvalue problem, where ω is the natural frequency and $\{W\}$ is the natural mode of vibration.

This algorithm was used to develop a program for calculating the natural frequencies and modes of vibrations for an arbitrary orthotropic shell of revolution of low shear rigidity which is partially filled with an ideal fluid.

Natural Vibrations of Shells Filled with Fluid. To test the approach proposed here and the program developed, we calculated vibrations of a full-scale shell with a fluid. The dependence of the lowest natural frequencies of shell vibrations ω on the fluid level was determined. The first natural vibration frequencies correspond to modes with the formation of four, three, five and six waves in the circumferential direction and one half-wave in the longitudinal direction (the first modes of the 4th, 3rd, 5th, and 6th circumferential harmonics). The calculation results for a shell with a fluid were compared with the experimental results of [5] on the natural frequencies and modes of small vibrations of a cylindrical shell of radius 0.1 m, length 0.58 m, and constant thickness $h = 0.0006$ m which was set upright and clamped at both ends. The shell was made of OT4 titanium alloy with an elastic modulus $E = 0.98 \cdot 10^5$ MPa, a density $\rho = 4550$ kg/m^3, and Poisson's constant $\nu = 0.3$. During the experiment, the service water level in the shell was varied from 0 to 100%.

Figures 4 and 5 gives curves of the natural shell vibration frequencies versus the relative fluid level η in the shell for various circumferential harmonics. As one can see from the curves, the calculation results are in good qualitative and satisfactory quantitative agreement with the experimental data. Some excess of the calculated frequencies over the experimental data can be explained by the fact that complete clamping of the shell ends is difficult to implement in practice.

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REFERENCES

1. A. Tessler and E. Saether, "A computationally viable higher-order theory for laminated composite plates," *Int. J. Numer. Method Eng.*, **31**, 1069–1086 (1991).
2. E. P. Kligman, "Natural vibrations of shells of revolution with finite shear rigidity," in: *Boundary-Value Problems of Elastic and Inelastic Systems* (collected scientific papers) [in Russian], Ural Scientific Center, USSR Academy of Sciences, Sverdlovsk (1985), pp. 91–97.
3. I. M. Rapoport, *Vibrations of an Elastic Shell Partially Filled with Fluid* [in Russian], Mashinostroenie, Moscow (1967).
4. D. H. Norrie and G. De Vries, *An Introduction to Finite-Element Analysis*, Academic Press, New York (1978).
5. V. F. Sivak and F. I. Telalov, "Experimental study of the vibrations of a cylindrical shell in contact with fluid," *Prikl. Mekh.*, **27**, No. 5, 63–68 (1991).